Discrete Applied Mathematics 160 (2012) 664–672

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Linear coloring of sparse graphs ☆

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ARTICLE INFO

Article history:
Received 23 June 2011
Received in revised form 11 October 2011
Accepted 21 October 2011
Available online 21 November 2011

Keywords:
Linear coloring
Girth
Maximum degree
Maximum average degree

ABSTRACT

A *linear k-coloring* of a graph G is a proper k-coloring of G such that the graph induced by the vertices of any two color classes is a union of vertex-disjoint paths. The *linear chromatic number* $\mathrm{lc}(G)$ of a graph G is the smallest number k such that G has a linear k-coloring. In this paper, we prove that if G is a graph with maximum degree Δ and maximum average degree $mad(G) < \frac{14}{5}$, then $\mathrm{lc}(G) \leq \lceil \frac{\Delta}{2} \rceil + 2$. In particular, $\mathrm{lc}(G) \leq \lceil \frac{\Delta}{2} \rceil + 2$ if G is a planar graph with maximum degree Δ and girth $g(G) \geq 7$. This improves some known results in this direction, and further supports a conjecture recently proposed by D. Cranston and G. Yu, which states that $\mathrm{lc}(G) \leq \lceil \frac{\Delta}{2} \rceil + 2$ for every graph G with $mad(G) < 3$.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. A *planar* graph is a graph that can be drawn on the Euclidean plane so that its edges only meet at their ends. Any such particular drawing of a planar graph is called a *plane* graph. For a graph G, we use $V(G)$, $E(G)$, $|G|$ and $\Delta(G)$ to denote its vertex set, edge set, order and maximum degree, respectively. For a vertex $v \in V(G)$, let $N_G(v)$ and $d_G(v)$ (or simply $d(v)$) denote the set of neighbors and the degree of v in G, respectively. A k (k^- or k^+)-*vertex* is a vertex of degree k (at most k or at least k, resp).

A *linear k-coloring* of a graph G is a proper k-coloring of G such that the graph induced by the vertices of any two color classes is a union of vertex-disjoint paths. The *linear chromatic number* $\mathrm{lc}(G)$ of a graph G is the smallest number k such that G has a linear k-coloring. The *maximum average degree* of a graph G is defined to be $mad(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\right\}$. The *girth* of a graph G, denoted $g = g(G)$, is the length of a shortest cycle of G. The following proposition can be easily derived from Euler's formula.

Proposition 1. *If G is a planar graph with girth g, then $mad(G) < \frac{2g}{g-2}$.*

The concept of linear coloring was first introduced by Yuster [5], who constructed an infinite family of graphs such that $\mathrm{lc}(G) \geq C_1\Delta(G)^{\frac{3}{2}}$ for some constant C_1, and also proved $\mathrm{lc}(G) \leq C_2\Delta(G)^{\frac{3}{2}}$ for some constant C_2 and sufficient large $\Delta(G)$.

Clearly, $\mathrm{lc}(G) \geq \left\lceil \frac{\Delta}{2} \right\rceil + 1$. It is interesting to notice that a number of authors proved that a lot of *sparse graphs* (with $mad(G) < 3$) attain this trivial lower bound. We would like to list the latest results in this direction as follows.

☆ Supported by Natural Science Foundation of Zhejiang Province, China, Grant No. Y6090699, and partially supported by the Natural Science Foundation of China, Grant No. 10971198, Zhejiang Innovation Project Grant No. T200905.

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doi:10.1016/j.dam.2011.10.028

Theorem A. *Let G be a graph. Then* $\mathrm{lc}(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$ *if:*

(1) *G is a tree* [3];
(2) *G can be embedded into a surface of nonnegative characteristic, and* $\Delta \geq 13, g \geq 7$, *or* $\Delta \geq 9, g \geq 8$, *or* $\Delta \geq 7, g \geq 9$, *or* $\Delta \geq 5, g \geq 10$, *or* $\Delta \geq 3, g \geq 13$ [4];
(3) $mad(G) < \frac{12}{5}$ *and* $\Delta(G) \geq 3$ *(list version)* [1].

On the other hand, as shown by a number of authors, the upper bound $C_2\Delta(G)^{\frac{3}{2}}$ for $\mathrm{lc}(G)$ given by Yuster can be decreased nearly to the trivial lower bound for sparse graphs. Recently, Cranston and Yu conjecture that $\mathrm{lc}(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 2$ for every graph G with $mad(G) < 3$. Some known evidences supporting this conjecture are given below.

Theorem B. *Let G be a graph:*

(1) *If G is planar and* $g(G) \geq 8$, *then* $\mathrm{lc}(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 2$ [2].
(2) *If G is planar and* $g(G) \geq 6$, *then* $\mathrm{lc}(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 3$ [2].
(3) *If G is planar and* $g(G) \geq 5$, *then* $\mathrm{lc}(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 4$ *(list version)* [1].
(4) *If* $mad(G) < 3$ *and* $\Delta(G) \geq 9$, *then* $\mathrm{lc}(G) \leq \left\lceil \frac{\Delta}{2} \right\rceil + 2$ *(list version)* [1].

In this paper we give a new evidence supporting the conjecture of Cranston and Yu.

Theorem 1. *If G is a graph with* $mad(G) < \frac{14}{5}$, *then* $\mathrm{lc}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$.

Applying Proposition 1, one can easily get the following corollary from Theorem 1.

Corollary 1. *If G is a planar graph with* $g(G) \geq 7$, *then* $\mathrm{lc}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2$.

The proof of Theorem 1 consists of two parts. Namely, we shall prove Theorem 1 for graphs with $\Delta(G) \geq 5$ in Section 2, and $\Delta(G) \leq 4$ in Section 3.

2. Graphs with $\Delta \geq 5$ and $mad(G) < \frac{14}{5}$

Call a 3-vertex *good* if it has no 2-neighbor; *non-good* otherwise.

Let H be a proper subgraph of G. A linear coloring of H using colors from a color set $C = \{1, 2, \ldots, k\}$ is called a *partial* linear k-coloring of G. In a partial linear k-coloring of G, we define $C_2(v)$ to be the subset of C, in which, every color appears exactly twice on the neighborhood $N_H(v)$ of a vertex v. Here and later, a color set means a *multi-set* of colors, i.e. a color may appear more than once. Hence we often say that some color appears in a color set exactly twice, at least twice, etc. Let x be a 2-vertex with neighbors y and z. We define $D(x)$ to be the subset of C, in which, every color appears at least twice on $N_H(y) \cup N_H(z)$ when $H = G - x$.

Theorem 2.1. *Let* $M \geq 5$ *be a positive integer and G a graph with* $\Delta(G) \leq M$. *If* $mad(G) < \frac{14}{5}$, *then* $\mathrm{lc}(G) \leq \left\lceil \frac{M}{2} \right\rceil + 2$.

Proof. Suppose G is a counterexample to Theorem 2.1 with the fewest vertices. Let H be a proper subgraph of G. Clearly, $\Delta(H) \leq M$ and $mad(H) < \frac{14}{5}$. By the choice of G, $\mathrm{lc}(H) \leq \left\lceil \frac{M}{2} \right\rceil + 2$, while $\mathrm{lc}(G) \geq \left\lceil \frac{M}{2} \right\rceil + 3$. As observed in earlier papers [2], G is connected and has neither 1-vertex nor adjacent 2-vertices.

From now on, let c be a partial linear coloring of G using colors from $C = \left\{1, 2, \ldots, \left\lceil \frac{M}{2} \right\rceil + 2\right\}$. Note that $|C| \geq 5$ since $M \geq 5$. □

Claim 2.1. *Every* 2*-vertex x has at most one* 4^-*-neighbor in G.*

Suppose to the contrary that x has two 4^--neighbors y and z. Without loss of generality, we may assume that both y and z are 4-vertices. Let y_1, y_2 and y_3 be the neighbors of y and z_1, z_2, z_3 the neighbors of z other than x. By the choice of G, $H = G - x$ has a linear coloring c using colors from C. If we can show that the linear coloring c on H can be extended to G, then we get a contradiction proving Claim 2.1. We first consider the case $c(y) = c(z)$. In this case, we could choose a color from $C \setminus (\{c(y)\} \cup D(x))$ to color x (since $|C| \geq 5$), yielding a linear coloring of G using colors from C. We next consider the case $c(y) \neq c(z)$. In this case, we could choose a color from $C \setminus (\{c(y), c(z)\} \cup C_2(y) \cup C_2(z))$ to color x, yielding a linear coloring of G using colors from C.

Recall that a 2-vertex has no 2-neighbors. Call a 2-vertex *type I*, *type II* if it has a (in fact, exactly one by Claim 2.1) 3-, 4-neighbor, respectively; *type III*, otherwise.

Claim 2.2. *A* 3*-vertex has at most one* 2*-neighbor in G; see* [2].

Claim 2.3. *If a* 3*-vertex* v *has a* 2*-neighbor x, then at least one of the other two neighbors y and z of* v *is a* 4^+*-vertex in G.*

By Claim 2.2, both y and z are 3^+-vertices, suppose both y and z are 3-vertices. Let x' be the neighbor of x, y_1 and y_2 the neighbors of y, z_1 and z_2 the neighbors of z, other than v. By the choice of G, $H = G - x$ has a linear coloring c using colors from C. If $c(v) = c(x')$, then we could choose a color from $C \setminus (\{c(v)\} \cup D(x))$ to color x, since $|D(x) \cup \{c(v)\}| \leq \left\lfloor \frac{d_H(x')+2}{2} \right\rfloor + 1 = \left\lfloor \frac{M-1+2}{2} \right\rfloor + 1 = \left\lceil \frac{M}{2} \right\rceil + 1 < |C| = \left\lceil \frac{M}{2} \right\rceil + 2$. So we may assume that $c(v) \neq c(x')$. Without loss of generality, let $c(v) = 1$, $c(x') = 2$. If $A = C \setminus (\{1, 2\} \cup C_2(v) \cup C_2(x')) \neq \emptyset$, we could choose a color from A to color x. So $A = \emptyset$. It follows that $C_2(v) \cup C_2(x') = \left\{3, 4, \dots, \left\lceil \frac{M}{2} \right\rceil + 2\right\}$. Without loss of generality, assume that $c(y) = c(z) = 3$ and $C_2(x') = \left\{4, \dots, \left\lceil \frac{M}{2} \right\rceil + 2\right\}$. If $2 \notin \{c(y_1), c(y_2)\} \cup \{c(z_1), c(z_2)\}$, we then recolor v with 2 and color x with 1. Otherwise, $2 \in \{c(y_1), c(y_2)\}$ or $2 \in \{c(z_1), c(z_2)\}$. In this case, there is a color $\alpha \in \left\{4, \left\lceil \frac{M}{2} \right\rceil + 2\right\}$ that appears at most once in $\{c(y_1), c(y_2), c(z_1), c(z_2)\}$, hence we could recolor v with α and color x with 1.

We use $n_2(v)$ to denote the number of 2-neighbors of a vertex v in G.

Claim 2.4. *If v is a 5-vertex such that $n_2(v) = 4$ and all the four 2-neighbors of v are of type I, then the remaining neighbor of v is a 4^+-vertex in G.*

Let $v_1, \dots, v_4$ be all the 2-neighbors of v that are of type I (i.e., for $i = 1, \dots, 4$, the neighbor of v_i other than v is a 3-vertex). Suppose to the contrary that v_5, the remaining neighbor of v, is a 3-vertex. Let u_i ($i = 1, \dots, 4$) be the neighbors of v_i other than v. Let w_i and z_i be the two neighbors of u_i other than v_i for $i = 1, \dots, 4$. Let w_5 and z_5 be the neighbors of v_5 other than v. By the choice of G, $H = G - v_1$ has a linear coloring c using colors from C. If $c(v) = c(u_1)$, then we could choose a color from $C \setminus (\{c(v)\} \cup D(v_1))$ to color v_1. So we may assume $c(v) \neq c(u_1)$. Without loss of generality, assume that $c(v) = 1$, $c(u_1) = 2$. If $A = C \setminus (\{1, 2\} \cup C_2(u_1) \cup C_2(v)) \neq \emptyset$, then we could choose a color from A to color v_1. So $A = \emptyset$. It follows that $|C| = 5$ and $C_2(u_1) \cup C_2(v) = \{3, 4, 5\}$. Without loss of generality, we may assume that $c(w_1) = c(z_1) = 3$, $c(v_2) = c(v_3) = 4$, $c(v_4) = c(v_5) = 5$. If we can recolor v_2 with 2 or 3, yielding a linear coloring of H, then we could color v_1 with 4. So either $c(u_2) = 2$, $c(w_2) = c(z_2) = 3$ or $c(u_2) = 3$, $c(w_2) = c(z_2) = 2$. Similarly the same conclusion holds for triple (u_4, w_4, z_4). Now if $c(w_5) = c(z_5) = 2$, then we recolor v_2 with 1, v with 3 and then color v_1 with 4, we are done. Otherwise, we could recolor v_2 with 1, v with 2 and then color v_1 with 4, yielding a linear coloring of G using colors from C, a contradiction proving Claim 2.4.

Claim 2.5. *If v is a 5-vertex such that $n_2(v) = 5$, then v has at least four 2-neighbors of type* III.

Let $v_1, \dots, v_5$ be all the 2-neighbors of v, u_i the neighbor of v_i other than $v (i = 1, \dots, 5)$. Suppose to the contrary that v has at most three 2-neighbors of type III. Let v_1 and v_2 be two 2-neighbors of type I or type II. Without loss of generality, we may assume that both v_1 and v_2 are of type II. For $i = 1, 2$, let w_i, z_i and t_i be the neighbors of u_i other than v_i. By the choice of G, $H = G - v_1$ has a linear coloring c using colors from C. If $c(v) = c(u_1)$, then we could choose a color from $C \setminus (\{c(v)\} \cup D(v_1))$ to color v_1, yielding a linear coloring of G using colors from C. So we may assume $c(v) \neq c(u_1)$, without loss of generality, let $c(v) = 1$, $c(u_1) = 2$. If $A = C \setminus (\{1, 2\} \cup C_2(u_1) \cup C_2(v)) \neq \emptyset$, then we could choose a color from A to color v_1, yielding a linear coloring of G using colors from C, a contradiction. It follows that $|C| = 5$ and $C_2(u_1) \cup C_2(v) = \{3, 4, 5\}$. Without loss of generality, we may assume that $c(w_1) = c(z_1) = 3$, $c(v_2) = c(v_3) = 4$ and $c(v_4) = c(v_5) = 5$. If recoloring v_2 with 2 or 3 yields a linear coloring of H, then we could color v_1 with 4, yielding a linear coloring of G using colors from C, a contradiction. So recoloring v_2 with 2 or 3 yields a coloring of H that is not proper or proper but not linear. It follows that $c(u_2) = 2$ and $\{c(w_2), c(z_2), c(t_2)\} = \{3, 3, \alpha\}$ or $c(u_2) = 3$ and $\{c(w_2), c(z_2), c(t_2)\} = \{2, 2, \beta\}$. We first consider the case $c(u_2) = 2$ and $\{c(w_2), c(z_2), c(t_2)\} = \{3, 3, \alpha\}$. If recoloring v_2, v with 1, 3, respectively, yields no (3, 5)-bicolored cycle, then we could further color v_1 with 4, giving a linear coloring of G using colors from C. Otherwise, we may recolor v_2, v with 1, 2, respectively, yielding a linear coloring of H using colors from C. Now if coloring v_1 with 1 produces no (2, 1)-bicolored cycle, then we are done. Otherwise, $c(t_1) = 1$, in this case, we could color v_1 with 4, yielding a linear coloring of G using colors from C. As for the case $c(u_2) = 3$ and $\{c(w_2), c(z_2), c(t_2)\} = \{2, 2, \beta\}$, an argument as above can yield the desired conclusion.

Claim 2.6. *If v is a 6-vertex such that $n_2(v) = 6$, then v has at most four 2-neighbors of type I.*

Suppose $v_1, \dots, v_6$ are all the 2-neighbors of v, u_i is the neighbor of v_i ($i = 1, \dots, 6$) other than v. Suppose to the contrary that v has five 2-neighbors of type I. Let v_6 be the unique neighbor of v that is not of type I. For $i = 1, \dots, 5$, let w_i and z_i be the two neighbors of u_i other than v_i. By the choice of G, $H = G - v_1$ has a linear coloring c using colors from C. If $c(v) = c(u_1)$, then we could choose a color from $C \setminus (\{c(v)\} \cup D(v_1))$ to color v_1, yielding a linear coloring of G using colors from C. So we may assume $c(v) \neq c(u_1)$, without loss of generality, assume that $c(v) = 1$, $c(u_1) = 2$. If $A = C \setminus (\{1, 2\} \cup C_2(u_1) \cup C_2(v)) \neq \emptyset$, then we could choose a color from A to color v_1. So $A = \emptyset$. It follows that $|C| = 5$ and $C_2(u_1) \cup C_2(v) = \{3, 4, 5\}$. Without loss of generality, we may assume that $c(w_1) = c(z_1) = 3$, $c(v_3) = c(v_4) = 4$, $c(v_5) = 5$ and $\{c(v_2), c(v_6)\} = \{5, \alpha\}$ where $\alpha = 2$ or 3. If recoloring v_4 with 2 or 3 yields a linear coloring of H, then we could color v_1 with 4, yielding a linear coloring of G using colors from C, a contradiction. So recoloring v_2 with 2 or 3 yields a coloring of H that is not proper or proper but not linear. It follows that $c(u_4) = 2$ and $c(w_4) = c(z_4) = 3$ or $c(u_4) = 3$ and $c(w_4) = c(z_4) = 2$. Similarly, for $i = 3, 5$, $c(u_i) = 2$ and $c(w_i) = c(z_i) = 3$ or $c(u_i) = 3$ and $c(w_i) = c(z_i) = 2$. Now we could recolor v_4, v with 1, $\beta \in \{2, 3\} \setminus \{\alpha\}$ and then color v_1 with 4, yielding a linear coloring of G using colors from C, a contradiction proving Claim 2.6.

To complete our proof of this theorem, it suffices to derive a contradiction by a discharging procedure.

First, we define the *initial charge function* μ on $V = V(G)$ by letting $\mu(v) = d(v) - \frac{14}{5}$ for every $v \in V$. Since $mad(G) < \frac{14}{5}$, the sum of the initial charge is negative. Note that only 2-vertices have negative charge. If we can make suitable discharging rules to redistribute charges among vertices so that the *final charge* $\mu'(v)$ of every vertex $v \in V$ is nonnegative, then we get a contradiction completing the proof.

The needed discharging rules are as follows.

R1. Every 2-neighbor of type I, type II, type III of a 5^+-vertex v gets $\frac{11}{20}, \frac{1}{2}, \frac{2}{5}$ from v, respectively.

R2. Every 2-neighbor of a 3-, 4-vertex v gets $\frac{1}{4}, \frac{3}{10}$ from v, respectively.

R3. Every non-good 3-neighbor of a 4^+-vertex v gets $\frac{1}{20}$ from v.

Now we are going to show that $\mu'(v) \geq 0$ for all $v \in V$.

Let v be a 2-vertex. Suppose the neighbors of v are x and y with $d(x) \leq d(y)$. If $d(x) = 3$ or $d(x) = 4$, then $d(y) \geq 5$ by Claim 2.1. By R1 and R2, $\mu'(v) \geq -\frac{4}{5} + \frac{11}{20} + \frac{1}{4} = 0$ or $\mu'(v) \geq -\frac{4}{5} + \frac{1}{2} + \frac{3}{10} = 0$. Otherwise, $d(x) \geq 5$ and $d(y) \geq 5$, we have $\mu'(v) \geq -\frac{4}{5} + 2 \times \frac{2}{5} = 0$ by R1.

Let v be a 3-vertex. By Claim 2.2, v has at most one 2-neighbor. If v has no 2-neighbor, then $\mu'(v) = \mu(v) = \frac{1}{5}$. So we may assume that v has exactly one 2-neighbor x, i.e., v is non-good. By Claim 2.3, at least one of the other two neighbors of v is a 4^+-vertex. By R2 and R3, $\mu'(v) \geq \frac{1}{5} - \frac{1}{4} + \frac{1}{20} = 0$.

Let v be a 4-vertex. By R2, $\mu'(v) \geq \frac{6}{5} - 4 \times \frac{3}{10} = 0$.

Let v be a 5-vertex. If $n_2(v) \leq 3$, then $\mu'(v) \geq \frac{11}{5} - 3 \times \frac{11}{20} - 2 \times \frac{1}{20} > 0$ by R1 and R3. Assume that $n_2(v) \geq 4$. First suppose $n_2(v) = 4$. If all the four 2-neighbors of v are of type I, then by Claim 2.4, the remaining neighbor of v is a 4^+-vertex, hence $\mu'(v) \geq \frac{11}{5} - 4 \times \frac{11}{20} - 0 = 0$ by R1. If at most three of the 2-neighbors are of type I, then $\mu'(v) \geq \frac{11}{5} - 3 \times \frac{11}{20} - \frac{1}{2} - \frac{1}{20} = 0$ by R1 and R3. Next suppose $n_2(v) = 5$. By Claim 2.5, v has at least four 2-neighbors of type III, hence $\mu'(v) \geq \frac{11}{5} - \frac{11}{20} - 4 \times \frac{2}{5} > 0$ by R1.

Let v be a 6-vertex. If $n_2(v) \leq 5$, then $\mu'(v) \geq \frac{16}{5} - \frac{11}{20} \times 5 - \frac{1}{20} > 0$ by R1 and R3. If $n_2(v) = 6$, then v has at most four 2-neighbors of type I by Claim 2.6, hence $\mu'(v) \geq \frac{16}{5} - 4 \times \frac{11}{20} - 2 \times \frac{1}{2} = 0$ by R1.

Finally let $d(v) \geq 7$. By R1, $\mu'(v) \geq d(v) - \frac{14}{5} - \frac{11}{20} \times d(v) = \frac{9d(v)-56}{20} > 0$.

3. Graphs with $\Delta \leq 4$ and $mad(G) < \frac{14}{5}$

It is trivial that $\mathrm{lc}(G) \leq \lceil \frac{\Delta}{2} \rceil + 2$ when $\Delta = \Delta(G) \leq 2$. By Theorem 2.1, to complete the proof of Theorem 1, we only need to prove the following.

Theorem 3.1. *If G is a graph with $3 \leq \Delta(G) \leq 4$ and $mad(G) < \frac{14}{5}$, then* $\mathrm{lc}(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 2 = 4$.

Proof. Suppose G is a counterexample to Theorem 3.1 with the fewest vertices, i.e., G is a graph such that $3 \leq \Delta(G) \leq 4$, $mad(G) < \frac{14}{5}$, $\mathrm{lc}(G) > 4$, but for any proper subgraph H of G, $\mathrm{lc}(H) \leq 4$. As noted before, G is connected and has neither 1-vertex nor adjacent 2-vertices. □

Claim 3.1. *Every 3-vertex has at most one 2-neighbor in G; see* [2].

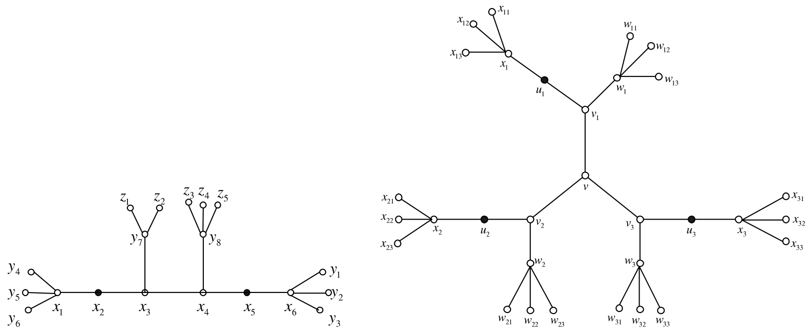
An (α, β)-*bicolored cycle (path)* is a cycle (path) consisting of vertices that are colored using colors α and β alternatively.

Claim 3.2. *There is no path $P = x_1x_2x_3x_4x_5x_6$ such that $d(x_2) = d(x_5) = 2$, $d(x_3) = d(x_4) = 3$, and x_3 has a 3-neighbor other than x_2 and x_4 in G.*

Suppose G contains such a path. Let y_1, y_2, y_3 be the neighbors of x_6 other than x_5, y_4, y_5, y_6 the neighbors of x_1 other than x_2, y_7 the neighbor of x_3 other than x_2 and x_4, y_8 the neighbor of x_4 other than x_3 and x_5, z_1 and z_2 the neighbors of y_7 other than x_3, z_3, z_4, z_5 the neighbors of y_8 other than x_4; see Fig. 1(a). By the choice of G, $H = G - x_5$ has a linear coloring c using colors from $C = \{1, 2, 3, 4\}$. If $c(x_4) = c(x_6)$, then we could choose a color from $C \setminus (\{c(x_4)\} \cup D(x_5))$ to color x_5, yielding a linear coloring of G using colors from C. So we may assume that $c(x_4) = 1$ and $c(x_6) = 2$. If $A = C \setminus (\{1, 2\} \cup C_2(x_6) \cup C_2(x_4)) \neq \emptyset$, then we could choose a color from A to color x_5, yielding a linear coloring of G using colors from C, a contradiction. Otherwise, $A = \emptyset$, i.e., $C_2(x_6) \cup C_2(x_4) = \{3, 4\}$. Without loss of generality, we may assume that $c(x_3) = c(y_8) = 3$, $c(y_1) = c(y_2) = 4$. If there is a color α that belongs to $\{2, 4\}$ and appears at most once in $\{c(x_2), c(y_7), c(z_3), c(z_4), c(z_5)\}$, then we could recolor x_4 with α and color x_5 with 1. Otherwise, each of 2 and 4 appears at least twice in $\{c(x_2), c(y_7), c(z_3), c(z_4), c(z_5)\}$. There are two cases under consideration.

(1) $\{c(x_2), c(y_7)\} = \{2, 4\}$.

First observe that $2, 4 \in \{c(z_3), c(z_4), c(z_5)\}$. Without loss of generality, we may assume that $c(z_3) = 2$ and $c(z_4) = 4$. There are two subcases (a) $c(x_2) = 2$, $c(y_7) = 4$, (b) $c(x_2) = 4$, $c(y_7) = 2$ under consideration. Since the proofs for (a) and (b) are rather similar, we only prove (a) as follows.



(a) The forbidden configuration in Claim 3.2.

(b) The forbidden configuration in Claim 3.3.

Fig. 1. The forbidden configurations in Claims 3.2 and 3.3.

We first suppose $c(z_5) \neq 2$. If recoloring x_4 with 2 produces no (2, 3)-bicolored cycle, then we could color x_5 with 1, yielding a linear coloring of G using colors from C. Otherwise, recoloring x_4 with 2 produces a (2, 3)-bicolored cycle. It follows that $c(x_1) = 3$, and exactly one of $c(y_4), c(y_5), c(y_6)$ is 2. Observe that there is a color $\beta \in \{1, 4\}$ that appears at most once in $\{c(y_4), c(y_5), c(y_6), c(y_7)\} \setminus \{2\}$. Now we could recolor x_2, x_4 with β, 2, respectively, and then color x_5 with 1, yielding a linear coloring of G using colors from C.

We next suppose $c(z_5) = 2$. If recoloring x_4 with 4 produces no (4, 3)-bicolored cycle, then we could color x_5 with 1. Otherwise, recoloring x_4 with 4 produces a (4, 3)-bicolored cycle, hence $3 \in \{c(z_1), c(z_2)\}$. Since c is a linear coloring of H, exactly one of $c(z_1), c(z_2)$ is 3. Now we could recolor x_3, x_4 with 1, 4, respectively, and then color x_5 with 1, yielding a linear coloring of G using colors from C.

(2) $c(x_2) = c(y_7)$.

There are two subcases (a) $c(x_2) = 2$, (b) $c(x_2) = 4$ under consideration. Since the proofs for (a) and (b) are rather similar, we only prove (a) as follows.

Clearly 4 appears exactly twice in $\{c(z_3), c(z_4), c(z_5)\}$. If $4 \notin \{c(z_1), c(z_2)\}$, then we could recolor x_3 with 4 and color x_5 with 3. So we may assume that $4 \in \{c(z_1), c(z_2)\}$.

We first suppose $(c(z_1), c(z_2)) \neq (4, 4)$. If recoloring x_3 with 4 produces no (4, 2)-bicolored cycle, then we could recolor x_3 with 4 and then color x_5 with 3. Otherwise, that recoloring x_3 with 4 produces a (4, 2)-bicolored cycle implies $c(x_1) = 4$. Observe that there is a color $\gamma \in \{1, 3\}$ that appears at most once in $\{c(y_4), c(y_5), c(y_6)\}$. Now we could recolor x_2, x_3 with γ, 4, respectively, and then color x_5 with 3, yielding a linear coloring of G using colors from C.

We next suppose $(c(z_1), c(z_2)) = (4, 4)$. Observe that $c(x_1) \in \{1, 3, 4\}$ (since $c(x_2) = 2$). If $c(x_1) = 1$, then there is a color $\alpha \in \{3, 4\}$ that appears at most once in $\{c(y_4), c(y_5), c(y_6)\}$, hence we could recolor x_2, x_3, x_4 with α, 1, 2, respectively, and then color x_5 with 1. If $c(x_1) = 3$, then there is a color $\beta \in \{1, 4\}$ that appears at most once in $\{c(y_4), c(y_5), c(y_6)\}$, hence we could recolor x_2, x_4 with β, 2, respectively, and then color x_5 with 1. Finally consider $c(x_1) = 4$. If 3 appears at most once in $\{c(y_4), c(y_5), c(y_6)\}$, then we could recolor x_2, x_3, x_4 with 3, 1, 2, respectively, and then color x_5 with 1. If 3 appears (exactly) twice in $\{c(y_4), c(y_5), c(y_6)\}$, then we could recolor x_2, x_4 with 1, 2, respectively, and then color x_5 with 1, yielding a linear coloring of G using colors from C, a contradiction proving Claim 3.2.

Call a 3-vertex *good* if it has no 2-neighbor; *non-good* otherwise. Observe that a non-good vertex is a 3-vertex with exactly one 2-neighbor by Claim 3.1.

Let v be a non-good vertex with two 3^+-neighbors v_1 and v_2. Call v *worst, worse, bad* if $(d(v_1), d(v_2)) = (3, 3), (3, 4), (4, 4)$, respectively.

Claim 3.3. *There is no* 3*-vertex v that has three non-good* 3*-neighbors in G.*

Suppose to the contrary that a 3-vertex v has three non-good 3-neighbors, say v_1, v_2 and v_3. For $i = 1, 2, 3$, let u_i be the unique 2-neighbor of v_i; w_i the neighbor of v_i other than v and u_i; w_{i1}, w_{i2} and w_{i3} the neighbor of w_i other than v_i; x_i the neighbor of u_i other than v_i; and x_{i1}, x_{i2} and x_{i3} the neighbor of x_i other than u_i; see Fig. 1(b). By the choice of G, $H = G - v$ has a linear coloring c using colors from C. There are three possible cases under consideration: (1) $c(v_1), c(v_2)$ and $c(v_3)$ are distinct; (2) $c(v_1), c(v_2)$ and $c(v_3)$ are identical; (3) exactly two of $c(v_1), c(v_2)$ and $c(v_3)$ are identical.

First suppose (1) happens. We show that either we can directly obtain a linear coloring of G using colors from C, or we can manage to lead (1) to (3). Without loss of generality, we may assume that $c(v_i) = i, i = 1, 2, 3$. If coloring v with 4 produces a linear coloring of G, then we are done. Otherwise, at least one pair of $(c(u_1), c(w_1))$, $(c(u_2), c(w_2))$ and $(c(u_3), c(w_3))$ is $(4, 4)$. Without loss of generality, we may assume that $(c(u_1), c(w_1)) = (4, 4)$. Let α be the color that belongs to $\{2, 3\}$ and appears at most once in $\{c(w_{11}), c(w_{12}), c(w_{13})\}$. We first recolor v_1 with α. If this produces no $(4, \alpha)$-bicolored cycle, then we lead (1) to (3); otherwise, this produces a $(4, \alpha)$-bicolored cycle, but then recoloring u_1 with β leads (1) to (3), where β is the color that belongs to $C \setminus \{4, \alpha\}$ and appears at most once in $\{c(x_{11}), c(x_{12}), c(x_{13})\}$.

Next suppose (2) happens. We show that (2) can also be led to (3). Suppose $c(v_1) = c(v_2) = c(v_3)$. Without loss of generality, we may assume $c(v_1) = 1$. If we recolor v_1 with one of 2, 3, 4, yielding a linear coloring of H, then we leads

(2) to (3). Otherwise, we have, without loss of generality, either (a) $c(u_1) = 2$, $c(w_1) = 3$, and 4 appears exactly twice in $\{c(w_{11}), c(w_{12}), c(w_{13})\}$, or (b) $c(u_1) = c(w_1) = 2$, 3 appears exactly twice in $\{c(w_{11}), c(w_{12}), c(w_{13})\}$, 4 appears exactly once in $\{c(w_{11}), c(w_{12}), c(w_{13})\}$, and there exists a (2, 4)-bicolored path in H connecting u_1 and w_1. If (a) happens, then we recolor v_1, u_1 with $2, \alpha$, respectively, where α is the color that belongs to $\{1, 3, 4\} \setminus \{c(x_1)\}$ and appears at most once in $\{c(x_{11}), c(x_{12}), c(x_{13})\}$, leading (2) to (3). If (b) happens, then we recolor u_1, v_1 with β, 4, respectively, where β is the color that belongs to $\{1, 3\}$ and appears at most once in $\{c(x_{11}), c(x_{12}), c(x_{13})\}$, leading (2) to (3).

Finally suppose (3) happens. Without loss of generality, we may assume that $c(v_1) = 2$ and $c(v_2) = c(v_3) = 1$. Observe that, in this case, 3 and 4 are symmetric. There are two cases under consideration: (3.1) $(c(u_1), c(w_1))$ is neither (3, 3) nor (4, 4); (3.2) $(c(u_1), c(w_1))$ is either (3, 3) or (4, 4).

(3.1) Note that, by symmetry of colors 3 and 4, we only need to consider three subcases: (3.1.1) $\{c(u_1), c(w_1)\} = \{1, 1\}$; (3.1.2) $\{c(u_1), c(w_1)\} = \{3, 1\}$; (3.1.3) $\{c(u_1), c(w_1)\} = \{3, 4\}$. We first claim that $\{c(w_2), c(u_2), c(w_3), c(u_3)\} = \{3, 3, 4, 4\}$. If $\{c(w_2), c(u_2), c(w_3), c(u_3)\} \neq \{3, 3, 4, 4\}$, without loss of generality, assuming that 3 appears at most once in $\{c(w_2), c(u_2), c(w_3), c(u_3)\}$, we could directly color v with 3, proving our claim.

First suppose that, without loss of generality, $c(u_2) = c(w_2) = 3$, $c(u_3) = c(w_3) = 4$. If we could recolor u_3 with 2 or 3, producing a linear coloring of H, then we could color v with 4, we are done. Otherwise, either $c(x_3) = 2$ and 3 appears twice in $\{c(x_{31}), c(x_{32}), c(x_{33})\}$ or $c(x_3) = 3$ and 2 appears twice in $\{c(x_{31}), c(x_{32}), c(x_{33})\}$. In this case, we first recolor u_3 with 1 and then recolor v_3 with a color γ that belongs to $\{2, 3\}$ and appears at most once in $\{c(w_{31}), c(w_{32}), c(w_{33})\}$. If coloring v with 4 produces no $(4, \gamma)$-bicolored cycle, then we are done. Otherwise, $\gamma = 2$ and a (4, 2)-bicolored cycle C^* is produced. If $u_1 \in C^*$, then $c(u_1) = 4$, $c(x_1) = 2$ and 4 appears exactly once in $\{c(x_{11}), c(x_{12}), c(x_{13})\}$. At present we could further recolor u_1 with a color δ that belongs to $\{1, 3\}$ and appears at most once in $\{c(x_{11}), c(x_{12}), c(x_{13}), c(w_1)\} \setminus \{4\}$, yielding a linear coloring of G. If $w_1 \in C^*$, we repeat the argument above starting from u_2, yielding the desired conclusion.

Next suppose that $c(u_2) \neq c(w_2)$. Assume that $c(u_2) = 3$, $c(w_2) = 4$. If coloring v with 3 produces no (1, 3)-bicolored cycle, then we are done. Otherwise, $c(x_2) = 1$ and 3 appears exactly once in $\{c(x_{21}), c(x_{22}), c(x_{23})\}$. Now we could further recolor u_2 with a color α that belongs to $\{2, 4\}$ and appears at most once in $\{c(x_{21}), c(x_{22}), c(x_{23}), c(w_2)\} \setminus \{3\}$, yielding a linear coloring of G. Similar argument can prove the case $c(u_2) = 4$, $c(w_2) = 3$.

(3.2) Without loss of generality, we may assume that $(c(u_1), c(w_1)) = (3, 3)$. If recoloring u_1 with 1 or 4 produces a linear coloring of H, then we go back (3.1). Otherwise, either $c(x_1) = 1$ and 4 appears exactly twice in $\{c(x_{11}), c(x_{12}), c(x_{13})\}$ or $c(x_1) = 4$ and 1 appears exactly twice in $\{c(x_{11}), c(x_{12}), c(x_{13})\}$. In this case, we first recolor u_1 with 2, then recolor v_1 with a color β that belongs to $\{1, 4\}$ and appears at most once in $\{c(w_{11}), c(w_{12}), c(w_{13})\}$, obtaining a linear coloring of H. If $\beta = 4$, then we go back (3.1). So we may assume that $\beta = 1$.

Observation 1. *Now H has two linear colorings c and c', which are the same except on u_1 and v_1. More precisely, $c(u) = c'(u)$, $u \in V(H) \setminus \{u_1, v_1\}$, $(c(u_1), c(v_1)) = (3, 2)$ and $(c'(u_1), c'(v_1)) = (2, 1)$.*

If $c(u_2) = c(w_2)$, then arguing as before Observation 1 can yield a linear coloring c'' of H, which is different from c' only on u_2 and v_2, $c''(v_2) \neq 1$, and $c''(u_2) \neq c''(w_2)$, hence go back (3.1). So we may assume that $c(u_2) \neq c(w_2)$. Similarly $c(u_3) \neq c(w_3)$.

If 4 appears at most once in $\{c(u_2), c(w_2), c(u_3), c(w_3)\}$, then we could color v with 4, yielding a linear coloring of G. So we may assume that 4 appears exactly twice in $\{c(u_2), c(w_2), c(u_3), c(w_3)\}$ by the last paragraph. Suppose $c(u_2) = 4$. We color v with 4. If this produces no (4, 1)-bicolored cycle, then we are done. Otherwise, this produces a (4, 1)-bicolored cycle, implying that $c(x_2) = 1$ and 4 appears exactly once in $\{c(x_{21}), c(x_{22}), c(x_{23})\}$. Now we could recolor u_2 with a color α that belongs to $\{2, 3\}$ and appears at most once in $\{c(x_{21}), c(x_{22}), c(x_{23}), c(w_2)\} \setminus \{4\}$, destroying the (4, 1)-bicolored cycle, hence giving a linear coloring of G. Similarly $c(u_3) \neq 4$. To conclude, $c(w_2) = c(w_3) = 4$.

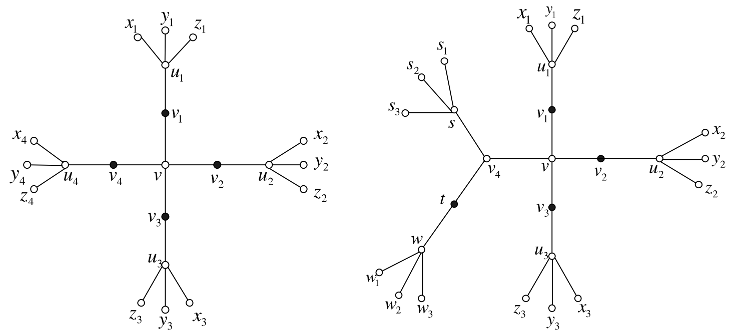
Now, let $c(u_2) = \alpha$. Clearly, $\alpha = 2$ or 3. If, based on c', recoloring v_2 with the color $\beta \in \{2, 3\} \setminus \{\alpha\}$ yields a new linear coloring of H, then we go back (3.1). Otherwise, β appears twice in $\{c(w_{21}), c(w_{22}), c(w_{23})\}$. Now we recolor v_2 with α. If we could further recolor u_2 with 1 or β, produces a linear coloring of H, then we go back (3.1) (based on c'). Otherwise, $c(x_2) = 1$ and β appears twice in $\{c(x_{21}), c(x_{22}), c(x_{23})\}$ or $c(x_2) = \beta$ and 1 appears twice in $\{c(x_{21}), c(x_{22}), c(x_{23})\}$. Now, based on c', we recolor u_2 with 4, and then color v with β. If this produces no $(\beta, 1)$-bicolored cycle, then we are done. Otherwise, this produces a $(\beta, 1)$-bicolored cycle C_2 that pass through u_3, implying $c(u_3) = \beta$, $c(x_3) = 1$ and β appears exactly once in $\{c(x_{31}), c(x_{32}), c(x_{33})\}$. Now we could recolor u_3 with a color that belongs to $\{\alpha, 4\}$ and appears at most once in $\{c(x_{31}), c(x_{32}), c(x_{33}), c(w_3)\} \setminus \{\beta\}$, destroying C_2, hence yielding a linear coloring of G. This proves Claim 3.3.

Claim 3.4. *There is no 4-vertex v that has four 2-neighbors in G.*

Suppose to the contrary that v has four 2-neighbors, say v_1, v_2, v_3, v_4. For $i = 1, \ldots, 4$, let u_i be the neighbor of v_i other than v, and x_i, y_i, z_i the neighbors of u_i other than v_i; see Fig. 2(a). By the choice of G, $H = G - v_1$ has a linear coloring c using colors from C. There are two cases under consideration.

(1) $c(v) = c(u_1) = 1$.

If $A = C \setminus (\{1\} \cup D(v_1)) \neq \emptyset$, then we could choose a color from A to color v_1, yielding a linear coloring of G using colors from C, a contradiction. Otherwise, $A = \emptyset$, i.e., $D(v_1) = \{2, 3, 4\}$. Observe that there are two possible cases: either $\{c(x_1), c(y_1), c(z_1)\} = \{2, 3, 4\}$ or $\{c(x_1), c(y_1), c(z_1)\} \neq \{2, 3, 4\}$.



(a) The forbidden configuration in Claim 3.4.

(b) The forbidden configuration in Claim 3.5.

Fig. 2. The forbidden configurations in Claims 3.4 and 3.5.

We first assume that $\{c(x_1), c(y_1), c(z_1)\} = \{2, 3, 4\}$. In this case, we may assume that $c(v_i) = i, i = 2, 3, 4$. If coloring v_1 with 2 produces no $(1, 2)$-bicolored cycle, then we are done. So we may assume that coloring v_1 with 2 produces a $(1, 2)$-bicolored cycle. It follows that $c(u_2) = 1$, and exactly one of $c(x_2)$, $c(y_2)$ and $c(z_2)$ is 2. If 4 appears at most once in $\{c(x_2), c(y_2), c(z_2)\}$, then we could recolor v_2, v with 4, 2, respectively, and then color v_1 with 3, yielding a linear coloring of G using colors from C. Otherwise, $\{c(x_2), c(y_2), c(z_2)\} = \{2, 4, 4\}$, then we could recolor v_2 with 3 and then color v_1 with 2, yielding a linear coloring of G using colors from C.

We next assume that $\{c(x_1), c(y_1), c(z_1)\} \neq \{2, 3, 4\}$. Without loss of generality, we may assume that $c(x_1) = c(y_1) = 2$, $c(z_1) = c(v_2) = 3$, $c(v_3) = c(v_4) = 4$. If coloring v_1 with 3 produces no $(1, 3)$-bicolored cycle, then we are done. So we may assume that coloring v_1 with 3 produces a $(1, 3)$-bicolored cycle, hence $c(u_2) = 1$. If recoloring v with 2 produces no $(2, 4)$-bicolored cycle, then we could further color v_1 with 3, yielding a linear coloring of G using colors from C. So we may assume that coloring v with 2 produces a $(2, 4)$-bicolored cycle. It follows that $c(u_3) = 2$ and exactly one of $c(x_3)$, $c(y_3)$ and $c(z_3)$ is 4. Observe that there is a color $\beta \in \{1, 3\}$ that appears at most once in $\{c(x_3), c(y_3), c(z_3)\}$. Now we could recolor v_3, v with β, 2, respectively, and then color v_1 with 4, yielding a linear coloring of G using colors from C.

(2) $c(v) \neq c(u_1)$.

Without loss of generality, we may assume that $c(v) = 1$, $c(u_1) = 2$. If $A = \{3, 4\} \setminus (C_2(u_1) \cup C_2(v)) \neq \emptyset$, then we could choose a color from A to color v_1, yielding a linear coloring of G using colors from C. So we may assume that $A = \emptyset$, i.e., $C_2(u_1) \cup C_2(v) = \{3, 4\}$. Without loss of generality, we may assume that $c(x_1) = c(y_1) = 3$, $c(v_2) = c(v_3) = 4$, and $c(v_4) = \alpha \in \{2, 3\}$.

If recoloring v with a color $\beta \in \{2, 3\} \setminus \{\alpha\}$ produces no $(\beta, 4)$-bicolored cycle, we could further color v_1 with 1, yielding a linear coloring of G using colors from C. So we may assume that recoloring v with β produces a $(\beta, 4)$-bicolored cycle, hence $c(u_2) = \beta$ and exactly one of $c(x_2)$, $c(y_2)$ and $c(z_2)$ is 4. Without loss of generality, assume that $c(x_2) = 4$, $c(y_2) = \gamma_1$, $c(z_2) = \gamma_2$. First suppose $(\gamma_1, \gamma_2) = (1, 1)$, then we could recolor v_2, v with α, β, respectively, and then color v_1 with 1, yielding a linear coloring of G using colors from C. Next suppose $(\gamma_1, \gamma_2) \neq (1, 1)$, then we could recolor v_2, v with 1, β, respectively, and then color v_1 with 1 if $\alpha = 2$; color v_1 with 4 or 1 according to $c(z_1) = 1$ or $c(z_1) \neq 1$ if $\alpha = 3$, yielding a linear coloring of G using colors from C.

Claim 3.5. *There is no* 4*-vertex v that has three* 2*-neighbors and a non-good* 3*-neighbor in G.*

Suppose to the contrary that v has three 2-neighbors v_1, v_2, v_3 and one 3-neighbor v_4. For $i = 1, 2, 3$, let u_i be the neighbor of v_i other than v; x_i, y_i, z_i the neighbor of u_i other than v_i. Let s and t be the neighbors of v_4 other than v, where $d(t) = 2$; s_1, s_2, s_3 the neighbors of s other than v_4, w the neighbor of t other than v_4; w_1, w_2, w_3 the neighbors of w other than t; see Fig. 2(b). By the choice of G, $H = G - v_2$ has a linear coloring c using colors from C. There are two cases under consideration.

(1) $c(v) = c(u_2) = 1$.

If $A = C \setminus (\{1\} \cup D(v_2)) \neq \emptyset$, then we could choose a color from A to color v_2, yielding a linear coloring of G using colors from C. So we may assume that $A = \emptyset$, i.e., $D(v_2) = \{2, 3, 4\}$. Observe that there are two possible subcases: either $\{c(x_2), c(y_2), c(z_2)\} = \{2, 3, 4\}$ or $\{c(x_2), c(y_2), c(z_2)\} \neq \{2, 3, 4\}$.

(1.1) We first assume that $\{c(x_2), c(y_2), c(z_2)\} = \{2, 3, 4\}$. In this case, we may assume that $c(v_1) = 2$, $c(v_3) = 3$, and $c(v_4) = 4$. If coloring v_2 with 4 produces no $(1, 4)$-bicolored cycle, then we are done. So we may assume that coloring v_2 with 4 produces a $(1, 4)$-bicolored cycle. It follows that exactly one of $c(s)$ and $c(t)$ is 1. If coloring v_2 with 2 produces no $(1, 2)$-bicolored cycle, then we are done. So we may assume that coloring v_2 with 2 produces a $(1, 2)$-bicolored cycle. It follows that $c(u_1) = 1$, and exactly one of $c(x_1)$, $c(y_1)$ and $c(z_1)$ is 2. Observe that there is a color $\alpha \in \{3, 4\}$ that appears at most once in $\{c(x_1), c(y_1), c(z_1)\}$. Now we could recolor v_1, v with α, 2, respectively, and then color v_2 with a color from $\{3, 4\} \setminus \{\alpha\}$, yielding a linear coloring of G.

(1.2) We next assume that $\{c(x_2), c(y_2), c(z_2)\} \neq \{2, 3, 4\}$. Without loss of generality, we may assume that $c(x_2) = c(y_2) = 2$, $c(z_2) = 3$. Namely $\{c(v_1), c(v_3), c(v_4)\} = \{3, 4, 4\}$.

(1.2.1) We first suppose that $c(v_4) = 3$, $c(v_1) = c(v_3) = 4$. If coloring v_2 with 3 produces no (3, 1)-bicolored cycle, then we are done. So coloring v_2 with 3 produces a (3, 1)-bicolored cycle. It follows that exactly one of $c(s)$ and $c(t)$ is 1. If recoloring v with 2 produces no (2, 4)-bicolored cycle, then we color v_2 with 3, yielding a linear coloring of G. Otherwise, recoloring v with 2 produces a (2, 4)-bicolored cycle. It follows that $c(u_3) = 2$ and exactly one of $c(x_3)$, $c(y_3)$ and $c(z_3)$ is 4. If 1 appears at most once in $\{c(x_3), c(y_3), c(z_3)\}$, then we could recolor v_3, v with 1, 2, respectively, and then color v_2 with 3. Otherwise, $\{c(x_3), c(y_3), c(z_3)\} = \{1, 1, 4\}$, then we could recolor v_3, v with 3, 2, respectively, and then color v_2 with 4.

(1.2.2) We next suppose that $c(v_3) = 3$, $c(v_4) = c(v_1) = 4$. If coloring v_2 with 3 produces no (3, 1)-bicolored cycle, then we are done. So coloring v_2 with 3 produces a (3, 1)-bicolored cycle. It follows that $c(u_3) = 1$ and exactly one of $c(x_3)$, $c(y_3)$ and $c(z_3)$ is 3.

(1.2.2.1) Color 2 appears at most once in $\{c(s), c(t)\}$.

If recoloring v with 2 produces no (2, 4)-bicolored cycle, then we color v_2 with 3, yielding a linear coloring of G. Otherwise, recoloring v with 2 produces a (2, 4)-bicolored cycle. It follows that $c(u_1) = 2$ and exactly one of $c(x_1)$, $c(y_1)$ and $c(z_1)$ is 4. Observe that there is a color $\beta \in \{1, 3\}$ that appears at most once in $\{c(x_1), c(y_1), c(z_1)\}$. Now we could recolor v_1, v with β, 2, respectively, and then color v_2 with 4, yielding a linear coloring of G.

(1.2.2.2) $c(s) = c(t) = 2$.

(a) $c(w) \in \{1, 3\}$. In this case, $\gamma \in \{1, 3\} \setminus \{c(w)\}$ appears exactly twice in $\{c(w_1), c(w_2), c(w_3)\}$ since otherwise we could recolor t with γ, going back (1.2.2.1). Now if 3 appears at most once in $\{c(s_1), c(s_2), c(s_3)\}$, then we could recolor t, v_4 with 4, 3, respectively, and then color v_2 with 4, yielding a linear coloring of G. Otherwise, 3 appears twice in $\{c(s_1), c(s_2), c(s_3)\}$, then we could recolor t, v_4, v with 4, 1, 2, respectively, and then color v_2 with 4, yielding a linear coloring of G.

(b) $c(w) = 4$. First 3 appears twice in $\{c(w_1), c(w_2), c(w_3)\}$ since otherwise we could recolor t with 3, going back (1.2.2.1). Next $1 \in \{c(w_1), c(w_2), c(w_3)\}$ since otherwise we could recolor t with 1, going back (1.2.2.1). Thus $\{c(w_1), c(w_2), c(w_3)\} = \{1, 3, 3\}$. Now if recoloring t with 1 produces no (1, 4)-bicolored cycle, then we go back (1.2.2.1). Otherwise, recoloring t with 1 produces a (1, 4)-bicolored cycle. It follows that $c(u_1) = 1$ and exactly one of $c(x_1)$, $c(y_1)$, $c(z_1)$ is 4. In this case, if 2 appears at most once in $\{c(x_1), c(y_1), c(z_1)\}$, then we can recolor t, v_1 with 1, 2, respectively, and then color v_2 with 4, yielding a linear coloring of G. Otherwise, $\{c(w_1), c(w_2), c(w_3)\} = \{4, 2, 2\}$, then we can recolor t, v_1 with 1, 3, respectively, and then color v_2 with 4, yielding a linear coloring of G.

(2) $c(v) \neq c(u_1)$.

Without loss of generality, we may assume that $c(v) = 1$ and $c(u_2) = 2$. If $A = \{3, 4\} \setminus (C_2(u_2) \cup C_2(v)) \neq \emptyset$, then we could choose a color from A to color v_2, yielding a linear coloring of G using colors from C. So we may assume that $A = \emptyset$, i.e., $C_2(u_2) \cup C_2(v) = \{3, 4\}$. Without loss of generality, we may assume that $c(x_2) = c(y_2) = 3$. There are two subcases under consideration.

(2.1) $c(v_3) = c(v_4) = 4$.

Clearly $c(v_1) = \alpha \in \{2, 3\}$. Let $\beta = \{2, 3\} \setminus \{\alpha\}$.

(2.1.1) Color β appears at most once in $\{c(s), c(t)\}$. If recoloring v with β produces no $(\beta, 4)$-bicolored cycle, then we can color v_2 with 1, yielding a linear coloring of G. Otherwise, recoloring v with β produces a $(\beta, 4)$-bicolored cycle. It follows that $c(u_3) = \beta$ and exactly one of $c(x_3)$, $c(y_3)$ and $c(z_3)$ is 4. If 1 appears at most once in $\{c(x_3), c(y_3), c(z_3)\}$, then we can recolor v_3, v with 1, β, respectively, and then color v_2 with 1 if $\alpha = 2$; color v_2 with 4 or 1 according to $c(z_2) = 1$ or $c(z_2) \neq 1$ if $\alpha = 3$, yielding a linear coloring of G. Otherwise, $\{c(x_3), c(y_3), c(z_3)\} = \{1, 1, 4\}$, then we can recolor v_3, v with α, β, respectively, and then color v_2 with 1, yielding a linear coloring of G.

(2.1.2) $c(s) = c(t) = \beta$.

(a) $c(w) \in \{1, \alpha\}$. In this case, $\gamma \in \{1, \alpha\} \setminus \{c(w)\}$ appears twice in $\{c(w_1), c(w_2), c(w_3)\}$ since otherwise we could recolor t with γ, going back (2.1.1). Now if α appears at most once in $\{c(s_1), c(s_2), c(s_3)\}$, then we could recolor t, v_4 with 4, α, respectively, and then color v_2 with 4, yielding a linear coloring of G. Otherwise, α appears twice in $\{c(s_1), c(s_2), c(s_3)\}$, then we could recolor t, v_4, v with 4, 1, β, respectively, and then color v_2 with 1 if $\alpha = 2$; color v_2 with 4 or 1 according to $c(z_2) = 1$ or $c(z_2) \neq 1$ if $\alpha = 3$, yielding a linear coloring of G.

(b) $c(w) = 4$. First α appears twice in $\{c(w_1), c(w_2), c(w_3)\}$ since otherwise we could recolor t with α, going back (2.1.1). Next $1 \in \{c(w_1), c(w_2), c(w_3)\}$ since otherwise we could recolor t with 1, going back (2.1.1). Thus $\{c(w_1), c(w_2), c(w_3)\} = \{1, \alpha, \alpha\}$. Now if recoloring t with 1 produces no (1, 4)-bicolored cycle, then we go back (2.1.1). Otherwise, recoloring t with 1 produces a (1, 4)-bicolored cycle. It follows that $c(u_3) = 1$ and exactly one of $c(x_3)$, $c(y_3)$ and $c(z_3)$ is 4. In this case, if β appears at most once in $\{c(x_3), c(y_3), c(z_3)\}$, then we can recolor t, v_3 with 1, β, respectively, and then color v_2 with 4, yielding a linear coloring of G. Otherwise, $\{c(w_1), c(w_2), c(w_3)\} = \{4, \beta, \beta\}$, then we can recolor t, v_3 with 1, α, respectively, and then color v_2 with 4, yielding a linear coloring of G.

(2.2) $c(v_3) = c(v_1) = 4$.

Clearly $c(v_4) = \alpha \in \{2, 3\}$. Let $\beta \in \{2, 3\} \setminus \{\alpha\}$.

(2.2.1) Color β appears at most once in $\{c(s), c(t)\}$. If recoloring v with β produces no $(4, \beta)$-bicolored cycle, then we could color v_2 with 1. Otherwise, recoloring v with β produces a $(4, \beta)$-bicolored cycle. It follows that $c(u_3) = \beta$ and exactly one of $c(x_3)$, $c(y_3)$ and $c(z_3)$ is 4. If 1 appears at most once in $\{c(x_3), c(y_3), c(z_3)\}$, then we recolor v_3, v with 1, β, respectively, and then we color v_2 with 1 if $\alpha = 2$; color v_2 with 4 or 1 according to $c(z_2) = 1$ or $c(z_2) \neq 1$ if $\alpha = 3$, yielding a linear coloring of G. Otherwise, $\{c(x_3), c(y_3), c(z_3)\} = \{1, 1, 4\}$. Now we can recolor v_3, v with α, β, respectively, and then color v_2 with 1, yielding a linear coloring of G.

(2.2.2) $c(s) = c(t) = \beta$.

(a) $c(w) = \alpha$. In this case, color 4 appears twice in $\{c(w_1), c(w_2), c(w_3)\}$ since otherwise we could recolor t with 4, going back (2.2.1). Now we can recolor t with 1, going back (2.2.1).

(b) $c(w) \in \{1, 4\}$. Observe that $\gamma \in \{1, 4\} \setminus \{c(w)\}$ appears twice in $\{c(w_1), c(w_2), c(w_3)\}$ since otherwise we could recolor t with γ, going back (2.2.1). In what follows, we try to recolor t, v_4 and v so that we could color v_2, yielding a linear coloring of G. According to the frequency of 1 appearing in $\{c(s_1), c(s_2), c(s_3)\}$, we distinguish two possibilities as follows.

(b1) Color 1 appears at most once in $\{c(s_1), c(s_2), c(s_3)\}$. In this case, we first recolor t, v_4, v with α, 1, 3, respectively. If this produces no (3, 4)-bicolored cycle, then we can color v_2 with 1, yielding a linear coloring of G. So we may assume that this produces a (3, 4)-bicolored cycle, hence $c(u_3) = 3$ and exactly one of $c(x_3)$, $c(y_3)$ and $c(z_3)$ is 4. Now if 2 appears at most once in $\{c(x_3), c(y_3), c(z_3)\}$, then we could recolor v_3 with 2, and then color v_2 with 1; otherwise, $\{c(x_3), c(y_3), c(z_3)\} = \{2, 2, 4\}$, we could recolor v_3 with 1 and then color v_2 with 4.

(b2) Color 1 appears twice in $\{c(s_1), c(s_2), c(s_3)\}$. In this case, we first recolor t, v_4 with α, 4, respectively. If we can recolor v_3 with 2 or 3, then we go back (2.1.1). Otherwise, it is easy to deduce that $c(u_3) = 2$, color 3 appears twice in $\{c(x_3), c(y_3), c(z_3)\}$ or $c(u_3) = 3$, color 2 appears twice in $\{c(x_3), c(y_3), c(z_3)\}$. Now we recolor v_3 with 1, v with 3. If this produces no (3, 4)-bicolored cycle, then we could color v_2 with 1. Suppose this produces a (3, 4)-bicolored cycle, hence $\alpha = 2$, $c(u_1) = 3$ and exactly one of $c(x_1)$, $c(y_1)$ and $c(z_1)$ is 4. If 2 appears at most once in $\{c(x_1), c(y_1), c(z_1)\}$, then we could further recolor v_1 with 2 and then color v_2 with 1. Otherwise, $\{c(x_1), c(y_1), c(z_1)\} = \{2, 2, 4\}$, then we can further recolor v_1 with 1 and then color v_2 with 4, yielding a linear coloring of G. Claim 3.5 is proved.

To complete our proof of this theorem, as in Section 2, we shall proceed a discharging procedure in G, yielding a contradiction. As before, the *initial charge function* μ on V still is $\mu(v) = d(v) - \frac{14}{5}$ for each vertex $v \in V$. Since the discharging procedures for $\Delta = 3, 4$ are rather similar but the former is easier, we only handle the later in detail.

The needed discharging rules for $\Delta = 4$ are as follows.

R1. Every 2-neighbor of a 3^+-vertex v gets $\frac{2}{5}$ from v.

R2. Every worst 3-neighbor of a good 3-vertex v gets $\frac{1}{10}$ from v.

R3. Every worse 3-neighbor, bad 3-neighbor of a 4-vertex v gets $\frac{1}{5}$, $\frac{1}{10}$ from v, respectively.

Now we are going to show that $\mu'(v) \geq 0$ for all $v \in V$.

Let v be a 2-vertex. By R1, $\mu'(v) \geq -\frac{4}{5} + 2 \times \frac{2}{5} = 0$.

Let v be a 3-vertex. By R1 and R2, v only gives charge to its 2-neighbor and worst 3-neighbor.

Let x, y, z be the three neighbors of v with $d(x) \leq d(y) \leq d(z)$. By Claim 3.1, at most one of x, y, z has degree 2. First suppose none of x, y and z has degree 2. If at least one of x, y and z has degree 4, then $\mu'(v) \geq \frac{1}{5} - \frac{1}{10} \times 2 = 0$ by R2. So we may assume that $d(x) = d(y) = d(z) = 3$. By Claim 3.3, at most two of x, y and z are non-good. We have $\mu'(v) \geq \frac{1}{5} - \frac{1}{10} \times 2 = 0$ by R2. Next suppose exactly one of x, y and z has degree 2. Thus $d(x) = 2$, $3 \leq d(y) \leq d(z)$. If $(d(y), d(z)) = (4, 4)$, i.e., v is a bad 3-vertex, then $\mu'(v) \geq \frac{1}{5} - \frac{2}{5} + 2 \times \frac{1}{10} = 0$ by R1 and R3. If $(d(y), d(z)) = (3, 3)$, i.e., v is a worst 3-vertex, then both y and z are not adjacent to any 2-vertex by Claim 3.2, i.e., both y and z are good 3-vertices, hence $\mu'(v) \geq \frac{1}{5} - \frac{2}{5} + 2 \times \frac{1}{10} = 0$ by R1 and R2. Finally suppose $(d(y), d(z)) = (3, 4)$, i.e., v is a worse 3-vertex. Let y_1 and y_2 be the neighbors of y other than v. If one of y_1 and y_2 is a 2-vertex, then the other is a 4-vertex by Claim 3.2. So both y and v are worse 3-vertices, hence $\mu'(v) \geq \frac{1}{5} - \frac{2}{5} + \frac{1}{5} + 0 = 0$ by R1 and R3; otherwise, $d(y_1) \geq 3$ and $d(y_2) \geq 3$, i.e., y is a good 3-vertex, hence $\mu'(v) \geq \frac{1}{5} - \frac{2}{5} + \frac{1}{5} + 0 = 0$ by R1 and R3.

Let v be a 4-vertex. By Claim 3.4, v has at most three 2-neighbors, say v_1, v_2 and v_3. Let x be the remaining neighbor of v. If $d(x) = 4$, then $\mu'(v) \geq \frac{6}{5} - \frac{2}{5} \times 3 - 0 = 0$ by R1. If $d(x) = 3$, then the two neighbors of x other than v have degree at least 3 by Claim 3.5, i.e., x is a good 3-vertex, hence $\mu'(v) \geq \frac{6}{5} - \frac{2}{5} \times 3 - 0 = 0$ by R1 and R3.

As for the discharging procedure for $\Delta = 3$, we only need to use R1, R2 and Claims 3.1–3.3 (note that proofs of these three claims work well for $\Delta = 3$).

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